1010 Revision (Tutorial 5) (last week：重陽節翌日補假 10/10)

Sequence : MCT, Sandwich theorem. Examples : Iterated sequence.

Continuity of a function : f is said to be continuous at x_0 iff $\lim_{x\to x_0} f(x) = f(x_0)$.

$\Rightarrow$ $\lim_{x\to x_0} f(x)$ exists.

Intermediate value theorem : (Existence theorem)

If f is continuous on $[a,b]$, $f(a) \neq f(b)$, and c is between $f(a)$ and $f(b)$,
then $\exists\ \xi \in (a,b)$ s.t. $f(\xi) = c$.
(means that $f(a) < c < f(b)$ / $f(b) < c < f(a)$)

Some applications of Intermediate value Thm : ("A first course in real analysis" by M.H. Protter, C.B. Morrey)

(a) If f is a polynomial of odd degree, then $f(\xi) = 0$ for some $\xi \in \mathbb{R}$ and range $(f) = \mathbb{R}$.

(b) Let $f : x \mapsto a_m x^m + a_{m-1} x^{m-1} + \dots + a_0$ be a polynomial of even degree. If $a_m a_0 < 0$, then $f(x) = 0$ has at least two real roots.

Differentiability of a function : f is said to be differentiable at x_0 iff $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists

$\Rightarrow$ f is continuous at x_0

(equivalently, $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists)

Differentiation :

Product rule : $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

$(f(x)g(x)h(x))' = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$

Chain rule ; inverse function theorem, implicit function theorem ~~(application)~~ (e.g. Q4, Q5 in Assignment 3)

Higher derivative : Leibniz's rule (Q6 in Asg 3) : $(fg)^{(n)}(x) = \sum_{r=0}^{n} \binom{n}{r} f^{(r)}(x)\, g^{(n-r)}(x)$

Mean value theorem : (Existence theorem).

Let f be cont. on $[a,b]$ (with $a<b$), and differentiable on (a,b), then
$\exists\ \xi \in (a,b)$ s.t. $f'(\xi) = \frac{f(b)-f(a)}{b-a}$

Then,

① function f on (a,b) s.t. $f'(x) = 0\ \forall x \in (a,b)$ $\Rightarrow$ f is a constant function

② $0 \leq f'(x)$ on $\forall x \in (a,b)$ and f cont on $[a,b]$ $\Rightarrow$ $f(a) \leq f(b)$

①' If $F_1'(x) = F_2'(x)\ \forall x \in (a,b)$, then $\exists$ const C s.t. $F_1(x) = F_2(x) + C\ \forall x \in (a,b)$

③ L' Hospital Rule.

④ Differentiability at a point.

① Differentiability at a point : (Assignment 2 : showing f is differentiable at a point)

let $a<b$. let $c \in (a,b)$, if $f'(x)$ exists $\forall x \in (a,b)\setminus\{c\}$, f continuous at c,

and $\lim_{x\to c} f'(x)$ exists, then $f'(c) := \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists (i.e. f is differentiable at c)

Proof: For $x>c$, By MVT: $\frac{f(x)-f(c)}{x-c} = \frac{f'(\xi_x)(x-c)}{x-c}$ for some $\xi_x \in (c,x)$.

$= f'(\xi_x)$.

For $x<c$, By MVT, $\frac{f(x)-f(c)}{x-c} = \frac{f'(\tilde{\xi}_x)(x-c)}{x-c} = f'(\tilde{\xi}_x)$ for some $\tilde{\xi}_x \in (x,c)$

$\therefore \lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c^+} f'(\xi_x) = \lim_{x\to c} f'(x)$

Similarly, $\lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} f'(x)$ $\therefore \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists $= \lim_{x\to c} f'(x)$ #

Example (MVT) : (Elementary Analysis through examples and exercises - John Schmeelk) p.246)

(a) If the Polynomial $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0$, $a_n \neq 0$ with real coefficients

has n real roots, then all roots of $P_n', P_n'', \dots, P_n^{(n-1)}$ are real.

(b) The roots of $\frac{d^n}{dx^n}(x^2-1)^n$ are real and located in $(-1,1)$.

Proof (a): Case ①: All roots of P_n are simple ($\Leftrightarrow$ it has n distinct roots)

let $c_1 < c_2 < \dots < c_n$ be all roots of P_n.

By MVT, $P_n'(\xi_i)(c_{i+1}-c_i) = P_n(c_{i+1}) - P_n(c_i) = 0$ $\forall i=1,\dots,n-1$

$\exists \xi_i \in (c_i, c_{i+1})$ s.t.

$\therefore$ P_n' has $(n-1)$ distinct real roots. Then by Induction.

Case②: Let $c_1 < c_2 < \dots < c_k$ be all distinct roots of P_n and each roots are of order $l_1,\dots,l_k$ respectively i.e. $P_n(x) = a_n(x-c_1)^{l_1}\dots(x-c_k)^{l_k}$, $l_1+\dots+l_k = n$.

Then, there are at least $(k-1)$ distinct roots for P_n', none of them equal c_i.

Note c_i are roots of P_n' if $l_i \geq 2$ and in this case, c_i are of order $\geq l_i - 1$ in P_n'

$\therefore$ $n-1 \geq$ # roots of P_n' (counting order) $\geq (k-1) + \sum_{i=1}^{k}(l_i - 1)$

$= k-1 + n-k = n-1$

(b) $\frac{d}{dx}(x^2-1)^n$ has a root ξ_1 in $(-1,1)$ and has order $(n-1)$ for roots $1,-1$

not a proof

$\frac{d^2}{dx^2}(x^2-1)^n$ has a roots $-1<\xi_2<\xi_1<\xi_3<1$ and has order $(n-2)$ for roots $1,-1$

$\frac{d^3}{dx^3}(x^2-1)^n$ has a roots $-1<\xi_4<\xi_2<\xi_5 \dots <\xi_3<\xi_6<1$ and order $(n-3)$ for roots $1,-1$

$\vdots$

$\frac{d^n}{dx^n}(x^2-1)^n$ has n roots in $(-1,1)$ (by induction)